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Abstract

We investigate the equivalence of quantum mixed states under local unitary transformations. For a class of rank-2 mixed states, a sufficient and necessary condition of local equivalence is obtained by giving a complete set of invariants under local unitary transformations, such that two states in this class are locally equivalent if and only if all these invariants have equal values for them.

PACS numbers: 03.67.−a, 02.20.Hj, 03.65.−w

Quantum entanglement has been extensively investigated as a key physical resource to realize quantum information tasks such as quantum cryptography, quantum teleportation and quantum computation [1]. Due to the fact that the properties of entanglement for multipartite quantum systems remain invariant under local unitary transformations on the subsystems, the entanglement can be characterized in principle by all the invariants under local unitary transformations. For instance, the trace norms of realigned or partial transposed density matrices in entanglement measure and separability criteria are some of these invariants [2]. Therefore, a complete set of invariants gives rise to the classification of the quantum states under local unitary transformations. Two quantum states are locally equivalent if and only if all these invariants have equal values for these states.

There have been many results on calculation of invariants [3, 4] related to the equivalence of quantum states under local unitary transformations, e.g. for general two-qubit systems [5], three-qubit states [6, 7], some generic mixed states [8–10], some classes of tripartite pure and mixed states [11]. However, till now, we still have no operational criteria to judge the equivalence for two general bipartite mixed states under local unitary transformations. In this paper, we investigate the local equivalence under local unitary transformations for a class

of rank-2 bipartite mixed quantum states in arbitrary dimensions, and present an operational criterion.

Let H_1 and H_2 be m- and n-dimensional complex Hilbert spaces, with $|e_\alpha\rangle, \alpha = 1, \ldots, m$, and $|f_\beta\rangle, \beta = 1, \ldots, n, m \leqslant n$, as orthonormal bases respectively. Let ρ_1 and ρ_2 be two bipartite density matrices defined on $H_1 \otimes H_2$ with rank $r(\rho_1) = r(\rho_2) = 2$. ρ_1 and ρ_2 are said to be equivalent under local unitary transformations if there exist unitary operators U_1 on H_1 and U_2 on H_2 such that

$$\rho_2 = (U_1 \otimes U_2)\rho_1(U_1 \otimes U_2)^\dagger, \tag{1}$$

where $\dagger$ stands for transpose and conjugation.

As ρ_1 and ρ_2 are rank-2 density matrices, they have the following decompositions according to their eigenvalues and eigenvectors:

$$\rho_i = \sum_{\alpha=1}^{2} \lambda_\alpha^i |v_\alpha^i\rangle\langle v_\alpha^i|, \qquad i = 1, 2,$$

where λ_α^i and $|v_\alpha^i\rangle, \alpha = 1, 2$, are the nonzero eigenvalues and eigenvectors of the density matrix ρ^i respectively, $\sum_{\alpha=1}^{2} \lambda_\alpha^i = 1$. $|v_\alpha^i\rangle$ generally has the form

$$|v_1^i\rangle = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{\alpha\beta}^i |e_\alpha\rangle \otimes |f_\beta\rangle, \qquad |v_2^i\rangle = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} b_{\alpha\beta}^i |e_\alpha\rangle \otimes |f_\beta\rangle,$$

where $a_{\alpha\beta}^i, b_{\alpha\beta}^i \in \mathbb{C}$, $\sum_{\alpha\beta} a_{\alpha\beta}^i a_{\alpha\beta}^{i*} = \sum_{\alpha\beta} b_{\alpha\beta}^i b_{\alpha\beta}^{i*} = 1$, $i = 1, 2$, $*$ denotes complex conjugation.

Let A_i and B_i denote the $m \times n$ matrices with entries $a_{\alpha\beta}^{(i)}$ and $b_{\alpha\beta}^{(i)}$ respectively. We consider the necessary and sufficient conditions of equivalence under local unitary transformations for a class of rank-2 states satisfying the following conditions:

$$A_i^\dagger A_i = B_i^\dagger B_i, \qquad A_i A_i^\dagger = B_i B_i^\dagger \quad \text{for} \quad i = 1, 2. \tag{2}$$

Theorem. *The density matrices ρ_1 and ρ_2 are equivalent under local unitary transformations if and only if the following hold:*

(i) $Tr(\rho_1^2) = Tr(\rho_2^2)$;
(ii) $Tr((A_1 B_1^\dagger)^\alpha) = Tr((A_2 B_2^\dagger)^\alpha), \forall \alpha = 1, \ldots, m$;
(iii) $r(A_1) = r(A_2), r(B_1) = r(B_2), r((B_1^\dagger A_1)^\alpha) = r((B_2^\dagger A_2)^\alpha), \forall \alpha = 1, \ldots, m$.

Proof. It is easy to see that (i)–(iii) above hold if ρ_1 and ρ_2 are equivalent under local unitary transformations, in the sense of equation (1).

We prove the converse. Two pairs of $(m \times n)$ matrices, (A, B) and (C, D), are called contragrediently equivalent if $A = SCT^{-1}, B = TDS^{-1}$ for some invertible matrices S and T. It is shown in [12] that the pairs (A, B) and (C, D) are contragrediently equivalent if and only if AB is similar to CD and $r(A) = r(C), r(B) = r(D), r(BA)^\alpha = r(DC)^\alpha, r(AB)^\alpha = r(CD)^\alpha$ for all $\alpha = 1, \ldots, m$.

Therefore from conditions (ii) and (iii), we have that the pairs $(A_1, B_1^\dagger)$ and $(A_2, B_2^\dagger)$ are contragrediently equivalent and there are invertible (but not necessarily unitary) matrices S and T such that

$$SA_2 = A_1 T, \qquad TB_2^{-1} = B_1^{-1} S. \tag{3}$$

Equation (3) can be rewritten as

$$\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}.$$

By assumption (2), the matrices $W_1 := \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix}$ and $W_2 := \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix}$ are normal.

If two normal matrices M, N and an invertible matrix X satisfy $X M X^{-1} = N$, then one has $U_X M U_X^\dagger = N$, where $X = U_X |X|$ is the polar decomposition of X and U_X is unitary [13]. Therefore from the observation that the unitary part of the polar decomposition in $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ is nothing but $\begin{pmatrix} 0 & U_S \\ U_T & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & U_T \\ U_S & 0 \end{pmatrix} \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & U_T \\ U_S & 0 \end{pmatrix},$$

which is equivalent to

$$A_2 = U_S^\dagger A_1 U_T, \qquad B_2 = U_S^\dagger B_1 U_T.$$

Here, U_S and U_T are unitary (as S, T are invertible). Condition (i) and $Tr(\rho_1) = Tr(\rho_2) = 1$ together imply that the density matrices ρ_1 and ρ_2 have the same eigenvalues. Therefore $\rho_2 = (U_1 \otimes U_2)\rho_1(U_1 \otimes U_2)^\dagger$, where $U_1 = U_S^\dagger$, $U_2 = (U_T)^t$ (t denoting the transpose). $\square$

The theorem gives a sufficient and necessary condition for local equivalence of two rank-2 mixed states satisfying (2). The class of quantum states satisfying (2) is not trivial. As a simple example, we consider the two-qubit systems. In this case, A and B are 2×2 matrices. It is easily verified that the following matrices satisfy the required conditions:

$$A(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad B(\gamma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}.$$

Hence the rank-2 density matrix $\rho = \lambda|\psi\rangle\langle\psi| + (1 - \lambda)|\phi\rangle\langle\phi|$, where $|\psi\rangle = \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}(\theta)|e_\alpha\rangle \otimes |f_\beta\rangle$, $|\phi\rangle = \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}(\gamma)|e_\alpha\rangle \otimes |f_\beta\rangle$, belongs to the class we are concerned with. From the theorem we have that ρ and $\rho' = \lambda|\psi'\rangle\langle\psi'| + (1 - \lambda)|\phi'\rangle\langle\phi'|$ with $|\psi'\rangle = \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}(\theta')|e_\alpha\rangle \otimes |f_\beta\rangle$, $|\phi'\rangle = \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}(\gamma')|e_\alpha\rangle \otimes |f_\beta\rangle$ are equivalent under local unitary transformations.

Here, the concurrence $C(|\psi\rangle) = C(|\phi\rangle) = 1$. Both pure states $|\psi\rangle$ and $|\phi\rangle$ are maximally entangled. In the special case $\theta = 0$ (respectively $\gamma = 0$), $|\psi\rangle$ (respectively $|\phi\rangle$) is reduced to one of the Bell bases $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (respectively $|\phi\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$). These states are equivalent under local unitary transformations. Nevertheless, generally ρ and ρ' are not equivalent under local unitary transformations even if $|\psi\rangle$ (respectively $|\phi\rangle$) is equivalent to $|\psi'\rangle$ (respectively $|\phi'\rangle$) under local unitary transformations, unless the same local unitary transformations transform $|\psi\rangle$ to $|\psi'\rangle$ and $|\phi\rangle$ to $|\phi'\rangle$ simultaneously.

Generally a rank-2 state can be written as $\rho = \lambda|v_1\rangle\langle v_1| + (1 - \lambda)|v_2\rangle\langle v_2|, 0 < \lambda < 1$. The normalized vectors $|v_1\rangle$ and $|v_2\rangle$ are given by the $m \times n$ matrices $(A)_{\alpha\beta} = a_{\alpha\beta}$ and $(B)_{\alpha\beta} = b_{\alpha\beta}$, respectively, $|v_1\rangle = \sum_{\alpha\beta} a_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$, $|v_2\rangle = \sum_{\alpha\beta} b_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$, with $Tr(AA^\dagger) = Tr(BB^\dagger) = 1$ due to normalization. Let us consider the general forms of a pair of matrices A and B such that the conditions $A^\dagger A = BB^\dagger$ and $AA^\dagger = BB^\dagger$ are satisfied.

Since $A^\dagger A = B^\dagger B$, we can write down singular value decomposition of A and B as follows:

$$A = U \Delta V^\dagger, \qquad B = U' \Delta V'^\dagger,$$

where U, U' and V, V' are unitary matrices and Δ is a diagonal matrix with non-negative entries. Furthermore, the condition $A^\dagger A = B^\dagger B$ implies $V'^\dagger V \Delta^2 = \Delta^2 V'^\dagger V$. Thus, $V'^\dagger V$

commutes with Δ. Similarly from $AA^\dagger = BB^\dagger$, we conclude that $U'^\dagger U$ also commutes with Δ. Hence, we have

$$B = U'\Delta V'^\dagger = UU^\dagger U'\Delta V'^\dagger VV^\dagger = U\Gamma\Delta V^\dagger, \tag{4}$$

where $\Gamma = (U'^\dagger U)^\dagger V'^\dagger V$ is unitary and commutes with Δ. Therefore, the pair (A, B) can be transformed into the pair $(\Delta, \Gamma\Delta)$. We call $(\Delta, \Gamma\Delta)$ the canonical form of the pair (A, B). If the diagonal matrix Δ is of the form $\mathrm{diag}(d_1, \ldots, d_1, d_2, \ldots, d_2, \ldots, d_k, \ldots, d_k)$, where d_i is repeated with multiplicity m_i, then Γ must have the block diagonal form $\mathrm{diag}(\Gamma_1, \ldots, \Gamma_k)$, where $\Gamma_i, i = 1, \ldots, k$, are $m_i \times m_i$ unitary matrices.

In fact, if we have another pair of matrices A' and B', associated with the eigenvectors $|\nu_1'\rangle$ and $|\nu_2'\rangle$ of another rank-2 density matrix in the class considered, with the canonical form $(\Delta', \Gamma'\Delta')$, then $|\nu_1'\rangle, |\nu_2'\rangle$ and $|\nu_1\rangle, |\nu_2\rangle$ are equivalent under local unitary transformations if and only if $\Delta = \Delta'$ and $w_i \Gamma_i w_i^\dagger = \Gamma_i'$ for some unitary matrix $w_i, i = 1, \ldots, k$.

Therefore under the local unitary transformation $\rho \to (U \otimes V^*)\rho(U \otimes V^*)^\dagger$, a rank-2 mixed state in our class has the standard form: $\rho = \lambda|\mu_1\rangle\langle\mu_1| + (1 - \lambda)|\mu_2\rangle\langle\mu_2|$, where $|\mu_1\rangle = \sum_\alpha d_\alpha|e_\alpha\rangle \otimes |f_\alpha\rangle$ and $|\mu_2\rangle = \sum_{\alpha\beta}(\Gamma\Delta)_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$. In particular, if all the singular values are distinct, then two such density matrices are equivalent under local unitary transformation if and only if they have exactly the same standard form.

We have investigated the equivalence under local unitary transformations for a class of rank-2 bipartite mixed quantum states. A complete set of invariants has been presented such that any two of these states are locally equivalent if and only if all these invariants have equal values for these related density matrices.

Our method can be applied to another classification of quantum states, defined by local operations and classical communication (LOCC). Two states have the same kind of entanglement if they can be obtained from each other by LOCC with nonzero probability [14]. There have been many results for bipartite and multipartite pure states for their equivalence under SLOCC [15–17]. In [15] Dür *et al* showed that for pure three-qubit states there are six different classes of entanglement under SLOCC. Verstraete *et al* [16] considered the entanglement of a four-qubit case under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement. Nevertheless, for mixed states few are known yet.

Corresponding to pure states, we say that ρ_1 and ρ_2 are equivalent under SLOCC if there exist *invertible* (but not necessarily unitary) matrices P and Q such that

$$\rho_2 = (P \otimes Q)\rho_1(P \otimes Q)^\dagger. \tag{5}$$

Proposition. *The density matrices ρ_1 and ρ_2, with B_1 and B_2 nonsingular, are equivalent under SLOCC if the following hold:*

(i) $Tr(\rho_1^2) = Tr(\rho_2^2)$;
(ii) $Tr((A_1 B_1^{-1})^\alpha) = Tr((A_2 B_2^{-1})^\alpha), \forall \alpha = 1, \ldots, m$;
(iii) $r(A_1) = r(A_2), r(B_1) = r(B_2), r((B_1^{-1}A_1)^\alpha) = r((B_2^{-1}A_2)^\alpha), \forall \alpha = 1, \ldots, m$.

Proof. From conditions (ii) and (iii), we have that the pairs (A_1, B_1^{-1}) and (A_2, B_2^{-1}) are contragrediently equivalent. Hence, there are invertible (but not necessarily unitary) matrices S and T such that $SA_2 = A_1 T, T B_2^{-1} = B_1^{-1} S$.

That is, we have $A_2 = S^{-1} A_1 T, B_2 = S^{-1} B_1 T$. Accounting to condition (i) which implies that the density matrices ρ_1 and ρ_2 have the same eigenvalues, the above relations give rise to the equivalence of ρ_1 and ρ_2 under SLOCC. $\qquad\square$

The classification of quantum states under local operations is of significance in quantum information processing. We have presented some criteria for the equivalence of some bipartite mixed states in arbitrary dimensions. Our results can be generalized to the case of multipartite states by considering bipartite decompositions. In terms of the method used in [11], our equivalence criteria for bipartite mixed states can also be used to study the equivalence of tripartite pure states.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Chen K, Albeverio S and Fei S M 2005 *Phys. Rev. Lett.* **95** 040504
   Chen K, Albeverio S and Fei S M 2003 *Phys. Rev.* A **68** 062313
[3] Rains E M 2000 *IEEE Trans. Inf. Theory* **46** 54–9
[4] Grassl M, Rötteler M and Beth T 1998 *Phys. Rev.* A **58** 1833
[5] Makhlin Y 2002 *Quantum Inf. Proc.* **1** 243
[6] Linden N, Popescu S and Sudbery A 1999 *Phy. Rev. Lett.* **83** 243
[7] Sun B Z and Fei S M 2006 *Commun. Theor. Phys.* **45** 1007–10
[8] Albeverio S, Fei S M, Parashar P and Yang W L 2003 *Phys. Rev.* A **68** 010303
[9] Albeverio S, Fei S M and Goswami D 2005 *Phys. Lett.* A **340** 37–42
[10] Sun B Z, Fei S M, Li-Jost X Q and Wang Z X 2006 *J. Phys. A: Math. Gen.* **39** L43–7
[11] Albeverio S, Cattaneo L, Fei S M and Wang X H 2005 *Int. J. Quantum Inf.* **3** 603–9
   Albeverio S, Cattaneo L, Fei S M and Wang X H 2005 *Rep. Math. Phys.* **56** 341–50
[12] Holtz O 2000 *Linear Algebr. Appl.* **310** 11–7
[13] Rudin W 1974 *Functional Analysis* (New Delhi: Tata Mcgraw-Hill)
[14] Bennett C H, Popescu S, Rohrlich D, Smolin J A and Thapliyal A V 2001 *Phys. Rev.* A **63** 012307
[15] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[16] Verstraete F, Dehaene J, De Moor B and Verschelde H 2002 *Phys. Rev.* A **65** 052112
[17] Acin A, Jane E, Dür W and Vidal G 2000 *Phys. Rev. Lett.* **85** 4811
   Briegel H J and Raussendorf R 2001 *Phy. Rev. Lett.* **86** 910